Approximation in Musielak-Orlicz sequence vector spaces of multifunctions

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Abstract. We introduce the space of vector multifunctions X_φ and we study its completeness. Also we give some approximation theorems in this space. Also we give some applications with singular kernel operators.

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1. Introduction

In [8] a general approximation theorem in modular spaces was obtained for linear operators. This theorem was extended in [2] and [9] to some nonlinear operators in L^φ and in [3, 4] to some simple space of multifunctions. In [7] these theorems were extended to vector spaces of multifunctions genereted by L^φ with Lebesgue measure. The aim of this note is to obtain a generalization of [8] and [5] for vector multifunctions with atom measure. Moreover in Section 5 we extend some results from [6, 9]. In Section 2 we give some initial informations about Musielak-Orlicz spaces and multifunctions.

Let $\mathbb{N}$ be the set of all nonnegative integers and l^φ be the Musielak-Orlicz sequence space generated by the modular

$$\rho(x) = \sum_{i=0}^{\infty} \varphi_i(t_i), \qquad x = (t_i),$$

where $\varphi = (\varphi_i)$ is a sequence of φ-functions with parameter , i.e. for every $i \in \mathbb{N}$ we have: $\varphi_i : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$, $\varphi_i(u)$ is an even continuous function, equal to zero iff

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$u = 0$ and nondecreasing for $u \geqslant 0$ with $\lim_{u\to\infty} \varphi_i(u) = \infty$. Let Y be a real separable Banach space with the norm $\|\cdot\|$. Let θ be the zero element in Y. Let $P_k(Y)$ denote the set of all nonempty and compact subsets of Y. For any $A, B \in P_k(Y)$ we denote

$$\operatorname{dist}(A,B) = \max\left\{\max_{x\in A}\min_{y\in B}\|x-y\|, \max_{y\in B}\min_{x\in A}\|x-y\|\right\},$$

$$X = \{F : \mathbb{N} \to 2^Y : F(i) \in P_k(Y) \text{ for every } i \in \mathbb{N}\}.$$

Every function from $\mathbb{N}$ to 2^Y will be called a sequential vector multifunction. For every $F \in X$ we define the functions $|F|$ by the formula:

$$|F|(i) = \operatorname{dist}(F(i), \theta) \qquad \text{for every } i \in \mathbb{N}.$$

Let now $[a, b]$ denote a compact interval for all $a, b \in \mathbb{R}$, $a \leqslant b$. Define

$$X_\varphi = \{F \in X : |F| \in l^\varphi\}.$$

Let $\mathbf{V}$ be an abstract set of indices . Let $\mathcal{V}$ be a filter of subsets of $\mathbf{V}$. Let $\mathbf{0} : \mathbb{N} \to Y$ be such that $\mathbf{0}(i) = \theta$ for every $i \in \mathbb{N}$.

2. Preliminary

We will present first some definitions and auxiliary results from the book [9].

Modular spaces

Definition 2.1. *Let X be a real vector space. A functional $\rho : X \to [0, +\infty]$ is called a modular, if the following conditions hold for arbitrary $x, y \in X$:*

1. *$\rho(0) = 0$ and $\rho(x) = 0$ implies $x = 0$,*
2. *$\rho(-x) = \rho(x)$,*
3. *$\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.*

If in place of 3 there holds

3'. $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$,

then the modular ρ is called convex.

Definition 2.2. *If ρ is a modular in X, then*

$$X_\rho = \{x \in X : \lim_{\lambda\to 0} \rho(\lambda x) = 0\}$$

is called a modular space.

Theorem 2.3. *If ρ is a modular in X, then*

$$|x|_\rho = \inf\{u > 0 : \rho(\frac{x}{u}) \leqslant u\}$$

is an F-norm in X_ρ, having the following properties:

- *if $\rho(\lambda x_1) \leqslant \rho(\lambda x_2)$ for every $\lambda > 0$, where $x_1, x_2 \in X_\rho$, then $|x_1|_\rho \leqslant |x_2|_\rho$ and moreover,*
- *if $x \in X_\rho$, then $|\lambda x|_\rho$ is a nondecreasing function of $\lambda \geqslant 0$,*
- *if $|x|_\rho < 1$, then $\rho(x) \leqslant |x|_\rho$.*

If ρ is a convex modular, then

$$\|x\|_\rho = \inf\{u > 0 : \rho(\frac{x}{u}) \leqslant 1\},$$

is a norm in X_ρ, which is called the Luxemburg norm.

Theorem 2.4. *Let ρ be a modular in X. If $x \in X_\rho$ and $x_k \in X_\rho$ for $k = 1, 2, \ldots$, then the condition $|x - x_k|_\rho \to 0$ as $k \to \infty$ is equivalent to the condition $\rho(\lambda(x - x_k)) \to 0$ as $k \to \infty$ for every $\lambda > 0$. If ρ is a convex modular in X, then the same statement holds, replacing $|\cdot|_\rho$ by $\|\cdot\|_\rho$.*

Definition 2.5. *Let ρ be a modular in X. A sequence $\{x_n\}$ of elements of X_ρ is called modular convergent to $x \in X_\rho$, if there exists a $\lambda > 0$ such that $\rho(\lambda(x_k - x)) \to 0$ as $k \to \infty$. We denote this writing $x_k \xrightarrow{\rho} x$.*

Theorem 2.6. *The ρ-convergence in X_ρ follows from norm convergence in X_ρ. Norm convergence and ρ-convergence are equivalent in X_ρ, if and only if, the following condition holds:*

$$\text{if } x_k \in X_\rho, \quad \rho(x_k) \to 0, \text{ then } \rho(2x_k) \to 0.$$

Definition 2.7. *Let ρ be a modular in X. A set $A \subset X_\rho$ will be called ρ-closed, if $x_k \in A$ and $x_k \xrightarrow{\rho} x$ imply $x \in A$.*

The smallest ρ-closed set containing the set A will be called the ρ-closure of A and denoted $\overline{A}^\rho$. If $\overline{A}^\rho = X_\rho$, then A will be called ρ-dense in X_ρ.

Musielak-Orlicz spaces

Definition 2.8. *Let (Ω, Σ, μ) be a measure space, where the measure μ is complete and not vanishing identically. A real function φ on $\Omega \times [0, +\infty)$, will be said to belong to the class Φ, if it satisfies the following conditions:*

i. *$\varphi(t, u)$ is a φ-function of the variable $u \geqslant 0$ for every $t \in \Omega$, i.e. $\varphi(t, u)$ is a nondecreasing, continuous function of u such that $\varphi(t, 0) = 0$, $\varphi(t, u) > 0$ for $u > 0$, $\varphi(t, u) \to \infty$ as $u \to \infty$,*
ii. *$\varphi(t, u)$ is a Σ-measurable function of t for every $u \geqslant 0$.*

Let X be the set of all real-valued, Σ-measurable and finite μ-almost everywhere functions on Ω, with equality μ-almost everywhere.

It is easily seen that $\varphi(t, |x(t)|)$ is a Σ-measurable function of t for every $x \in X$ and that

$$\rho(x) = \int_{\Omega} \varphi(t, |x(t)|) d\mu$$

is a modular in X. Moreover, if $\varphi(t, u)$ is a convex function of u for all $t \in \Omega$, then ρ is a convex modular in X.

Definition 2.9. *The modular space X_ρ will be called Musielak-Orlicz space and denoted by L^φ:*

$$L^\varphi = \{x \in X : \int_{\Omega} \varphi(t, \lambda |x(t)|) d\mu \to 0 \text{ as } \lambda \to 0_+\}.$$

Moreover, the set

$$L_0^\varphi = \{x \in X : \int_{\Omega} \varphi(t, |x(t)|) d\mu < \infty\}$$

will be called the Musielak-Orlicz class. A function $x \in X$ will be called a finite element of L^φ, if $\lambda x \in L_0^\varphi$ for every $\lambda > 0$. The space of all finite elements of X will be denoted by E^φ.

If X is the space of sequences $x = \{t_i\}$ with real terms t_i, $\varphi = \{\varphi_i\}$, where φ_i are φ-functions and

$$\rho(x) = \sum_{i=1}^{\infty} \varphi_i(|t_i|),$$

we shall write l^φ in place of L^φ and l^φ is called the Musielak-Orlicz sequence space.

Theorem 2.10.

a. *L^φ is the set of all $x \in X$ such that $\rho(\lambda x) < \infty$ for some $\lambda > 0$.*
b. *L_0^φ is a convex subset of L^φ and L^φ is the smallest vector subspace of X containing L_0^φ.*
c. *E^φ is the largest vector subspace of X contained in L_0^φ.*

Definition 2.11. *A function φ will be called locally integrable, if*

$$\int_A \varphi(t, u) d\mu < \infty$$

for every $u > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$.

Theorem 2.12. *Let S be the set of all simple, integrable functions on Ω and let $\varphi \in \Phi$ be locally integrable. Then $S \subset E^\varphi$. Moreover, supposing μ to be σ-finite, E^φ is the closure of S with respect to the F-norm $|\cdot|_\rho$ and S is ρ-dense in L^φ.*

Theorem 2.13. *Let μ be σ-finite. Then the Musielak-Orlicz space L^φ is complete with respect to the F-norm $|\cdot|_\rho$.*

Theorem 2.14.

a. *If μ is σ-finite and atomless and $\varphi \in \Phi$, then $E^\varphi = L_0^\varphi = L^\varphi$, if and only if, the following condition holds:*

$$(\Delta_2) \qquad \varphi(t, 2u) \leqslant K\varphi(t, u) + h(t)$$

for all $u \geqslant 0$ and almost every $t \in \Omega$, where h is a nonnegative, integrable function in Ω and K is a positive constant.

b. *If $\varphi \in \Phi$, where $\varphi = \{\varphi_n\}$, then $l_0^\varphi = l^\varphi$, if and only if, the following condition holds: there exist positive numbers δ, K and a sequence $\{a_n\}$ of nonnegative numbers with $\sum\limits_{n=1}^{\infty} a_n < \infty$ such that*

$$(\delta_2) \qquad \varphi_n(u) < \delta \text{ implies } \varphi_n(2u) \leqslant K\varphi_n(u) + a_n$$

for all $u \geqslant 0$, $n = 1, 2, \ldots$

Let us remark that the implication

$$(\Delta_2) \Rightarrow L_0^\varphi = L^\varphi$$

holds without any assumptions on the measure μ for any $\varphi \in \Phi$.

Theorem 2.15.

a. *If μ is σ-finite and atomless and $\varphi \in \Phi$, φ is locally integrable, then the following conditions are mutually equivalent:*
   (1) $L_0^\varphi = L^\varphi$,
   (2) $E^\varphi = L^\varphi$,
   (3) φ satisfies the condition (Δ_2),
   (4) modular convergence and norm convergence are equivalent in L^φ.

b. *If $\varphi \in \Phi$, where $\varphi = \{\varphi_n\}$, then the following conditions are mutually equivalent:*
   (1) $l_0^\varphi = l^\varphi$,
   (2) $E^\varphi = l^\varphi$,
   (3) $\varphi = (\varphi_i)$ satisfies the condition (δ_2),
   (4) modular convergence and norm convergence are equivalent in l^φ.

Let us remark that if $\varphi \in \Phi$ is a convex function of the variable $u \in R$ for every $t \in \Omega$, then φ is of the form

$$\varphi(t, u) = \int_0^{|u|} p(t, \tau) d\tau, \tag{1}$$

where $p(t, u)$ is the right-hand derivative of $\varphi(t, u)$ for a fixed $t \in \Omega$.

Definition 2.16. *We shall say that a function $\varphi \in \Phi$ is an N-function if φ is a convex function of u for every $t \in \Omega$ and there hold the conditions:*

$$\lim_{u\to 0_+} \frac{\varphi(t,u)}{u} = 0, \tag{2}$$

$$\lim_{u\to\infty} \frac{\varphi(t,u)}{u} = \infty \tag{3}$$

for every $t \in \Omega$.

Theorem 2.17. *$\varphi \in \Phi$ is an N-function, if and only if, φ is of the form (1), where $p(t,\tau) > 0$ for $\tau > 0$, $p(t,\tau)$ is a right-continuous and nondecreasing function of $\tau \geqslant 0$, $p(t,0) = 0$, $p(t,\tau) \to \infty$ as $\tau \to \infty$ for every $t \in \Omega$.*

Remark 2.18. Let φ be an N-function of the form (1) and let

$$p^\star(t,\sigma) = \sup\{\tau : p(t,\tau) \leqslant \sigma\}.$$

If p satisfies the conditions expressed in the previous theorem, then $p^\star$ satisfies the same assumptions.

Definition 2.19. *Let φ be an N-function of the form (1) and let $p^\star$ be defined by the formula*

$$p^\star(t,\sigma) = \sup\{\tau : p(t,\tau) \leqslant \sigma\}.$$

Then the function

$$\varphi^\star(t,u) = \int_0^{|u|} p^\star(t,\sigma)d\sigma$$

is called complementary to φ in the sense of Young.

Evidently, $\varphi^\star$ is again an N-function.

Theorem 2.20. *Let φ be an N-function and let $\varphi^\star$ be complementary to φ in the sense of Young. Then they satisfy the Young inequality*

$$uv \leqslant \varphi(t,u) + \varphi^\star(t,v)$$

for $u, v \geqslant 0$, $t \in \Omega$, and

$$\varphi^\star(t,v) = \sup_{u>0}\{uv - \varphi(t,u)\}, \qquad \varphi(t,u) = \sup_{v>0}\{uv - \varphi^\star(t,v)\},$$

consequently, φ is complementary to $\varphi^\star$ in the sense of Young.

We shall introduce now the Orlicz norm $\|\cdot\|_{\rho,O}$ in L^φ if φ is an N-function.

Theorem 2.21. *Let a measure μ be σ-finite, φ be an N-function and φ locally integrable, $\varphi^\star$ complementary to φ in the sense of Young and let*

$$L_1^{\varphi^\star} = \Big\{y : \int_\Omega \varphi^\star(t,|y(t)|)d\mu \leqslant 1, \quad y \text{ measurable}\Big\}.$$

Then

$$\|x\|_{\rho,O} = \sup_{y\in L_1^{\varphi^\star}} \int_\Omega x(t)y(t)d\mu$$

is a norm in L^{φ} *(called the Orlicz norm) and*

$$\|x\|_{\rho} \leqslant \|x\|_{\rho,O} \leqslant 2\|x\|_{\rho}$$

for all $x \in L^{\varphi}$.

Theorem 2.22. *Let* φ *be an* N*-function,* $\varphi^{\star}$ *complementary to* φ *in the sense of Young,* $x \in L^{\varphi}$, $y \in L^{\varphi^{\star}}$. *Then there hold the following Hölder inequalities:*

$$\Big|\int_{\Omega} x(t)y(t)d\mu\Big| \leqslant \|x\|_{\rho,O}\|y\|_{\rho^0},$$

$$\Big|\int_{\Omega} x(t)y(t)d\mu\Big| \leqslant \|x\|_{\rho}\|y\|_{\rho^0,O},$$

where

$$\rho(x) = \int_{\Omega} \varphi(t,|x(t)|)d\mu, \qquad \rho^0(y) = \int_{\Omega} \varphi^{\star}(t,|y(t)|)d\mu.$$

Corollary 2.23. *If* φ *is an* N*-function and* $\varphi^{\star}$ *is a complementary to* φ *in the sense of Young,* $y \in L^{\varphi^{\star}}$, *then*

$$f(x) = \int_{\Omega} x(t)y(t)d\mu$$

is a linear, continuous functional over L^{φ} *with the norm* $\|f\| = \|y\|_{\rho^0,O}$.

Definition 2.24. *We say that functional* $f : L^{\varphi} \to \mathbb{R}$ *is* ρ*- continuous if* $x_n \xrightarrow{\rho} 0$ *in* L^{φ} *implies* $f(x_n) \to 0$.

Theorem 2.25. *Let a measure* μ *be* σ*-finite and let an* N*-function* φ *satisfies the following condition:*

- *for every* $u_0 > 0$ *there exists a* $c > 0$ *such that* $\frac{\varphi(t,u)}{u} \geqslant c$ *for* $u \geqslant u_0$ *and all* $t \in \Omega$.

Let the function $\varphi^{\star}$ *complementary to* φ *be locally integrable. Then*

$$f(x) = \int_{\Omega} x(t)y(t)d\mu$$

is a ρ*-continuous linear functional over* L^{φ} *for every* $y \in L^{\varphi^{\star}}$.

Theorem 2.26. *Let a measure* μ *be* σ*-finite and let* N*-function* φ *be such that:*

- *for every* $u_0 > 0$ *there exists a* $c > 0$ *for which* $\frac{\varphi(t,u)}{u} \geqslant c$ *for* $u \geqslant u_0$ *and* $t \in \Omega$.

Moreover, let both functions φ *and* $\varphi^{\star}$ *(complementary to* φ*) be locally integrable. Then for every linear,* ρ*-continuous functional* f *over* L^{φ} *there exists a function* $y \in L^{\varphi^{\star}}$ *such that*

$$f(x) = \int_{\Omega} x(t)y(t)d\mu$$

for every $x \in L^{\varphi}$.

Theorem 2.27. *Let an N-function φ and its complementary $\varphi^\star$ be locally integrable. Let us suppose that for every $u_0 > 0$ there is a $c > 0$ for which $\frac{\varphi(t,u)}{u} \geqslant c$ for $u \geqslant u_0$ and $t \in \Omega$. Moreover, let us suppose that one of the following two conditions is satisfied:*

a. a measure μ is σ-finite and atomless and φ satisfies the condition (Δ_2),
b. $\Omega = \{1, 2, \ldots\}$, $\mu(\{n\}) = 1$ for $n = 1, 2, \ldots$ and condition (δ_2) holds for φ.

Then the general form of a linear functional over L^φ continuous with respect to the norm is

$$f(x) = \int\limits_\Omega x(t)y(t)d\mu$$

for $x \in L^\varphi$ with $y \in L^{\varphi^\star}$, and $\|f\| = \|y\|_{\rho^0}, 0$.

Now we will present some general definitions and auxiliary results about the multifunctions from the book [1].

Let now X be a Hausdorff topological space. Denote:

- 2^X: the collection of all subsets of X,
- $2^X \setminus \emptyset$: the collection of all nonempty subsets of X.
- $P_f(X) = \{A \subseteq X :$ nonempty, closed$\}$,
- $P_{fc}(X) = \{A \subseteq X :$ nonempty, closed, convex$\}$,
- $P_{bf}(X) = \{A \subseteq X :$ nonempty, bounded, closed$\}$,
- $P_{bfc}(X) = \{A \subseteq X :$ nonempty, bounded, closed, convex$\}$,
- $P_k(X) = \{A \subseteq X :$ nonempty, compact$\}$,
- $P_{kc} = \{A \subseteq X :$ nonempty, compact, convex$\}$.

If (X, d) is a metric space we denote:

$$B_X(x, a) = \{y \in X : d(x, y) < a\}.$$

Let now (X, d) be a metric space. In what follows given any $x \in X$ and $A \in 2^X \setminus \emptyset$, the distance of x from A, is defined by

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

As usual, $d(x, \emptyset) = +\infty$.

Definition 2.28. *If $A, C \in 2^X$, we define*

a. $h^\star(A, C) = \sup\{d(a, C) : a \in A\}$,
b. $h^\star(C, A) = \sup\{d(c, A) : a \in C\}$,
c. $h(A, C) = \max\{h^\star(A, C), h^\star(C, A)\}$, the Hausdorff distance between A and C.

Directly from the definition, we can check that the following properties hold for any $A, C, D \in 2^X$:

$$h(A, A) = 0, \quad h(A, C) = h(C, A), \quad h(A, C) \leqslant h(A, D) + h(D, C).$$

Hence h is an extended pseudometric on 2^X.

Moreover, note that $h(A, C) = 0$ if and only if $\overline{A} = \overline{C}$. So $P_f(X)$ equipped with the Hausdorff distance h becomes a metric space.

Theorem 2.29. *If (X, d) is a complete metric space, then so is $(P_f(X), h)$.*

Theorem 2.30. *If (X, d) is a complete metric space, then $P_k(X)$ is a closed subset of $(P_f(X), h)$, whence $(P_k(X), h)$ is a complete metric space.*

Theorem 2.31. *$P_{bf}(X)$ is a closed subset of $(P_f(X), h)$. Therefore, if (X, d) is a complete metric space, so is $(P_{bf}(X), h)$.*

Theorem 2.32. *If X is a Banach space, then $P_{kc}(X)$, $P_{bfc}(X)$, $P_{fc}(X)$, $P_k(X)$, $P_{bf}(X)$ are complete subspaces of the metric space $(P_f(X), h)$.*

Theorem 2.33. *If X is a normed space and $A, C, A_1, C_1, A_2, C_2 \in 2^X \setminus \emptyset$, then*

$$h(\lambda A, \lambda C) = |\lambda|\, h(A, C) \qquad \text{for all } \lambda \in R,$$

$$h(A_1 + A_2, C_1 + C_2) \leqslant h(A_1, C_1) + h(A_2 + C_2).$$

Measurable multifunctions

Now we will present some definitions and auxiliary results about the measurable multifunction from the book [1].

Throughout this section (Ω, Σ) is a measurable space, (X, d) a separable metric space.Let us fix a multifunction $F : \Omega \to 2^X$.

Definition 2.34.

a. F is said to be "strongly measurable" if for every $C \subseteq X$ closed, we have

$$F^-(C) = \{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \Sigma.$$

b. F is said to be "measurable" if for every $U \subseteq X$ open, we have

$$F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma.$$

c. F is said to be "K-measurable" if for every $K \subseteq X$ compact, we have

$$F^-(K) = \{\omega \in \Omega : F(\omega) \cap K \neq \emptyset\} \in \Sigma.$$

d. F is said to be "graph measurable" if

$$GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X).$$

Theorem 2.35. *If F is strongly measurable, then F is measurable.*

Theorem 2.36. *A multifunction $F : \Omega \to 2^X$ is measurable if and only if for every $x \in X$,*

$$\omega \to d(x, F(\omega)) = \inf\{d(x, x') : x' \in F(\omega)\}$$

is a measurable $\overline{R}_+$-valued function.

Definition 2.37. *Let Y be a metric space. A function $f : \Omega \times X \to Y$ is said to be a "Caratheodory function" if*

a. for every $x \in X$, $\omega \to f(\omega, x)$ is measurable,
b. for every $\omega \in \Omega$, $x \to f(\omega, x)$ is continuous.

Theorem 2.38. *If Y is a metric space and $f : \Omega \times X \to Y$ is a Caratheodory function, then $f(\cdot,\cdot)$ is jointly measurable.*

Theorem 2.39. *If $F : \Omega \to P_f(X) \cup \emptyset$ is measurable, then F is graph measurable.*

Theorem 2.40. *If $F : \Omega \to P_k(X)$, then F is strongly measurable if and only if it is measurable.*

Theorem 2.41. *If $F : \Omega \to P_f(X)$, then strong measurability $\Rightarrow$ measurability $\Rightarrow$ K-measurability.*

Theorem 2.42. *If X is σ-compact and $F : \Omega \to P_f(X)$, then strong measurability $\Leftrightarrow$ measurability $\Leftrightarrow$ K-measurability.*

Theorem 2.43. *Let (Ω, Σ, μ) be a σ-finite and complete measure space. Let (X, d) be complete separable metric space and $F : \Omega \to P_f(X)$. Consider the following statements:*

(a) for every $D \in B(X)$, $F^-(D) \in \Sigma$,
(b) F is strongly measurable,
(c) F is measurable,
(d) for every $x \in X$, $\omega \to d(x, F(\omega))$ is measurable,
(e) $GrF \in \Sigma \times B(X)$,

then all these statements are equivalent.

Theorem 2.44. *If (Ω, Σ) is a measurable space, X is a complete measure space and $F : \Omega \to P_f(X)$ is measurable, then F admits a measurable selection, i.e., there exists $f : \Omega \to X$ measurable such that for every $\omega \in \Omega$, $f(\omega) \in F(\omega)$.*

Theorem 2.45. *If (Ω, Σ) is a measurable space, X is a complete metric space and $F : \Omega \to P_f(X)$, then the following statements are equivalent:*

a. F is measurable,
b. there exists a sequence $\{f_n\}_{n\geqslant 1}$ of measurable selectors of F such that for every $\omega \in \Omega$

$$F(\omega) = \overline{\{f_n(\omega)\}}_{n\geqslant 1}.$$

Decomposable sets and sets of L^p selectors

Now we will present some definitions and auxiliary results about the decomposable sets and the sets of selectors from the book [1].

Let now (Ω, Σ, μ) be a σ-finite measure space, X a Banach separable space. Let $L^0(\Omega, X)$ be the space of all equivalent classes in the set of all measurable maps from Ω to X.

Definition 2.46. *A subset K of $L^0(\Omega, X)$ is said to be decomposable if for all $f_1, f_2 \in L^0(\Omega, X)$, $A \in \Sigma$, we have*

$$\chi_A f_1 + \chi_{\Omega \setminus A} f_2 \in K.$$

For $1 \leqslant p \leqslant \infty$, we define

$$S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \ \ \mu - a.e.\}.$$

Lemma 2.47. *If $F : \Omega \to 2^X \setminus \emptyset$ is graph measurable and $1 \leqslant p \leqslant \infty$, then $S_F^p \neq \emptyset$ if and only if*

$$\inf\{\|x\| : x \in F(\omega)\} \leqslant h(\omega) \ \ \mu - a.e.$$

for some $h \in L^p(\Omega)$.

Theorem 2.48. *If $F : \Omega \to P_f(X)$ is graph measurable and $S_F^p \neq \emptyset$, then there exists a sequence $\{f_n\}_{n\geqslant 1} \subseteq S_F^p$ such that*

$$F(\omega) = \overline{\{f_n\}}_{n\geqslant 1} \mu - a.e.$$

Theorem 2.49. *If K is a nonempty, closed subset of $L^p(\Omega, X)$ for $1 \leqslant p < \infty$, then $K = S_F^p$ for some uniquely defined measurable multifunction $F : \Omega \to P_f(X)$ if and only if K is decomposable.*

Definition 2.50. *A multifunction $F : \Omega \to 2^X \setminus \emptyset$ is said to be L^p-integrably bounded ($1 < p \leqslant \infty$) and integrably bounded (for $p = 1$) if there exists $h \in L^p(\Omega)$ such that*

$$|F(\omega)| := \sup\{\|x\| : x \in F(\omega)\} \leqslant h(\omega) \ \ \mu - a.e.$$

Theorem 2.51. *If F is graph measurable, then S_F^p is $L^p(\Omega, X)$-bounded if and only if F is L^p-integrably bounded ($1 < p \leqslant \infty$).*

Integral of multifunction

Now we will present some definitions and auxiliary results about the integral of multifunction from the book [1].

Throughout out this section (Ω, Σ, μ) is a fixed σ-finite measure space and X is a separable Banach space. Let $F : \Omega \to 2^X \setminus \emptyset$ be a multifunction with $S_F^1 \neq \emptyset$. Then the set-valued Aumann integral of F is defined in the following way.

Definition 2.52.

$$\int_\Omega F(\omega) d\mu(\omega) = \Big\{ \int_\Omega f(\omega) d\mu(\omega) : f \in S_F^1 \Big\}.$$

We say that two measurable multifunctions $F_1, F_2 : \Omega \to 2^X \setminus \emptyset$ are equivalent if $F_1(\omega) = F_2(\omega)$ μ-a.e. Denote by $\mathbf{L}_f^1(X)$ the space of all equivalence classes of multifunctions $F : \Omega \to P_f(X)$ which are graph measurable and integrably bounded. Also by $\mathbf{L}_{fc}^1(X)$ we donote the subspace of all (equivalence classes) of graph measurable and integrably bounded multifunctions with values in $P_{fc}(X)$. Since

$$h(F(\omega), G(\omega)) \leqslant |F(\omega)| + |G(\omega)|,$$

we deduce that $h(F, G) \in L^1(\Omega)_+$. So we can define

$$\Delta(F, G) = \int_{\Omega} h(F(\omega), G(\omega)) d\mu(\omega).$$

It is easily seen that Δ is a metric on $\mathbf{L}^1_f(X)$ and we have

Theorem 2.53. *The space* $(\mathbf{L}^1_f(X), \Delta)$ *is a complete metric space and* $(\mathbf{L}^1_{fc}(X), \Delta)$ *is its closed subspace.*

Theorem 2.54. *If* $F, G \in \mathbf{L}^1_f(X)$, *then*

$$h(\int_{\Omega} F(\omega) d\mu(\omega), \int_{\Omega} G(\omega) d\mu(\omega)) \leqslant \Delta(F, G).$$

Theorem 2.55. *If* $F, G : \Omega \to P_f(X)$ *are graph measurable with* $S^1_G, S^1_F \neq \emptyset$, *then*

$$cl \int_{\Omega} \overline{(F(\omega) + G(\omega))} d\mu(\omega) = cl[\int_{\Omega} F(\omega) d\mu(\omega) + \int_{\Omega} G(\omega) d\mu(\omega)].$$

Theorem 2.56. *If* $F : \Omega \to 2^X \setminus \emptyset$ *is a graph measurable multifunctions with* $S^1_F \neq \emptyset$, *then*

$$cl \int_{\Omega} \overline{conv} F(\omega) d\mu(\omega) = \overline{conv} \int_{\Omega} F(\omega) d\mu(\omega) = cl \int_{\Omega} conv F(\omega) d\mu(\omega).$$

Theorem 2.57. *If the measure* μ *is nonatomic,* $F : \Omega \to P_f(X)$ *is graph measurable and* $S^1_F \neq \emptyset$, *then* $cl \int_{\Omega} F(\omega) d\mu(\omega)$ *is convex.*

Corollary 2.58. *If* μ *is nonatomic,* X *is finite dimensional,* $F : \Omega \to P_f(X)$ *is graph measurable and* $S^1_f \neq \emptyset$, *then* $\int_{\Omega} F(\omega) d\mu(\omega)$ *is convex.*

Theorem 2.59. *If* μ *is nonatomic,* $F : \Omega \to P_f(R^n)$ *is graph measurable and for every* $\omega \in \Omega$, $F(\omega) \subseteq R^n_+$, *then*

$$\int_{\Omega} F(\omega) d\mu(\omega) = \int_{\Omega} conv F(\omega) d\mu(\omega).$$

3. Main results

We start from the generalization of the definition of Musielak-Orlicz sequence space of multifunctions from [5]. We use some ideas from [3, 5, 7, 8] and we generalize the main approximation theorem for l^{φ} from [8].

Theorem 3.1. *Let $F_n \in X_\varphi$ for every $n \in \mathbb{N}$. Suppose that for every $\epsilon > 0$ and for every $a > 0$ there is $K > 0$ such that $\rho(a\,\mathrm{dist}(F_n(\cdot), F_m(\cdot)) < \epsilon$ for all $m, n > K$. Then there exists $F \in X_\varphi$, such that $\rho(a\,\mathrm{dist}(F_n(\cdot), F(\cdot))) \to 0$ as $n \to \infty$ for every $a > 0$.*

Proof. Let $F_n \in X_\varphi$ for every $n \in \mathbb{N}$. If the assumptions of the Theorem hold, then $\{F_n\}$ is a Cauchy sequence in the complete space $C(Y)$ with Hausdorff metric. Hence there are $F(i) \in C(Y)$ such that $dist(F_n(i), F(i)) \to 0$ as $n \to \infty$ for every $i \in \mathbb{N}$. Fix $\epsilon > 0$. Applying the Fatou lemma we easily obtain that there exists $K > 0$ such that $\rho(a\,\mathrm{dist}(F_n(\cdot), F(\cdot)) \leqslant \epsilon$ for every $n > K$. We also have for every $a > 0$ that

$$\rho(a|F|) \leqslant \rho(2a\,\mathrm{dist}(F_n(\cdot), F(\cdot)) + \rho(2a|F_n|).$$

So $F \in X_\varphi$. □

The space X_φ will be called a Musielak-Orlicz vector sequence space of multifunctions.

Definition 3.2. *A function $g : \mathbf{V} \to R$ tends to zero with respect to a filter $\mathcal{V}$, written $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\epsilon > 0$ there is $V \in \mathcal{V}$ such that $\mid g(v) \mid < \epsilon$ for every $v \in V$.*

Definition 3.3. *An operator $C : X_\varphi \to X_\varphi$ will be called an X-linear operator if for all $F, G \in X_\varphi$, $a, b \in \mathbb{R}$,*

$$C(aF + bG)(i) = aC(F)(i) + bC(G)(i) \qquad \text{for every } i \in \mathbb{N}.$$

Definition 3.4. *A family $T = (T_v)_{v \in \mathbf{V}}$ of operators $T_v : X_\varphi \to X_\varphi$, for every $v \in \mathbf{V}$ will be called $(X, \mathrm{dist}, \mathcal{V})$-bounded, if there exist constants $k_1, k_2 > 0$ and a function $g : \mathbf{V} \to \mathbb{R}_+$ such that $g(v) \xrightarrow{\mathcal{V}} 0$, and for all $F, G \in X_\varphi$ there is a set $V_{F,G} \in \mathcal{V}$ for which*

$$\rho(a\,\mathrm{dist}(T_v(F)(\cdot), T_v(G)(\cdot))) \leqslant k_1 \rho(ak_2\,\mathrm{dist}(F(\cdot), G(\cdot))) + g(v)$$

for all $v \in V_{F,G}$ and for every $a > 0$.

Definition 3.5. *Let $F_v \in X_\varphi$ for every $v \in \mathbf{V}$. Let $F \in X_\varphi$. We write $F_v \xrightarrow{d,\varphi,\mathcal{V}} F$, if for every $\epsilon > 0$ and every $a > 0$ there exists $V \in \mathcal{V}$ such that $\rho(a\,\mathrm{dist}(F_v(\cdot), F(\cdot))) < \epsilon$ for every $v \in V$.*

Definition 3.6. *Let $S \subset X_\varphi$.*

$$S_{X_\varphi,d,\mathcal{V}} = \{F \in X_\varphi : F_v \xrightarrow{d,\varphi,\mathcal{V}} F, \text{ for some } F_v \in S, v \in \mathbf{V}\}.$$

Theorem 3.7. *Let the family $T = (T_v)_{v \in \mathbf{V}}$ of X-linear operators for every $v \in \mathbf{V}$, be $(X, \mathrm{dist}, \mathcal{V})$-bounded. Let $S_o \subset X_\varphi$ and let $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in S_o$. Let now S be the set of all finite linear combinations of elements of the set S_o. Then $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in S_{X_\varphi,d,\mathcal{V}}$.*

The proof analogous to that Theorem 4 in [5] is omitted.

4. Applications

Let now $\mathbf{V} = \mathbb{N}$ and the filter $\mathcal{V}$ will consist of all sets $V \subset \mathbf{V}$ which are complements of finite sets.

We shall say that φ is τ_+-bounded, if there are constants $k_1, k_2 \geqslant 1$ and a double sequence $\{\epsilon_{n,j}\}$ such that

$$\varphi_{n+j}(u) \leqslant k_1 \varphi_n(k_2 u) + \epsilon_{n,j}$$

for $u \in R, n, j = 0, 1, \ldots$, where $\epsilon_{n,j} \geqslant 0$, $\epsilon_{n,0} = 0$, $\epsilon_j = \sum\limits_{n=0}^{\infty} \epsilon_{n,j} < \infty$, $i \in \mathbb{N}$, and $\epsilon_j \to 0$

as $j \to \infty$, $\mathbf{s} = \sup_{j \in \mathbb{N}} \epsilon_j < \infty$. Let $K_{v,j} : \mathbf{V} \times \mathbf{V} \to \mathbb{R}_+$ and let the family $(K_v)_{v \in \mathbf{V}}$ be almost-singular, i.e. $\sigma(v) = \sum_{j=0}^{\infty} K_{v,j} \leqslant \sigma < \infty$ for all $v \in \mathbf{V}$ and $\frac{K_{v,j}}{\sigma(v)} \xrightarrow{\mathcal{V}} 0$ for $j = 1, 2, \ldots$ Let $F \in X_\varphi$. We define a family $\mathcal{T} = (\mathcal{T}_v)_{v \in \mathbf{V}}$ of operators by the formula:

$$\mathcal{T}_v(F)(i) = \sum_{j=0}^{i} K_{v,i-j} F(j) \text{ for every } i \in \mathbf{V}.$$

Lemma 4.1. *Let $(K_v)_{v \in \mathbf{V}}$ be almost-singular, let $\varphi = (\varphi_i)_{i \in \mathbf{V}}$ be τ_+-bounded and φ_i be convex for every $i \in \mathbf{V}$, then $\mathcal{T}_v : l^\varphi \to l^\varphi$ for every $v \in \mathbf{V}$.*

The proof analogous to that of Lemma 1 in [5] is omitted.

Lemma 4.2. *If the assumptions of Lemma 1 hold, then the family $\mathcal{T} = (\mathcal{T}_v)_{v \in \mathbf{V}}$ is $(X_\varphi, \mathrm{dist}, \mathcal{V})$-bounded and $\mathcal{T}_v$ is X_φ-linear-operator for every $v \in \mathbf{V}$.*

The proof analogous to that of Lemma 2 in [5] is omitted.

Lemma 4.3. *Let $\varphi = (\varphi_i)_{i=0}^{\infty}$ satisfy the condition (δ_2). Let $F \in X_\varphi$ and $F = (F(i))_{i=0}^{\infty}$. Let F_v be such that $F_v(i) = F(i)$ for $i = 0, 1, \ldots, v$ and $F_v(i) = 0$ for $i > v$. Then $F_v \xrightarrow{d,\varphi,\mathcal{V}} F$.*

The proof analogous to that of Lemma 3 in [5] is omitted.

Now, let us denote: $x_{j,K_v} = \{\underbrace{0, \ldots, 0}_{j-times}, K_{v,1}, K_{v,2}, \ldots\}$.

Theorem 4.4. *Let the assumptions of Lemmas 1 and 3 hold. If $x_{j,K_v} \xrightarrow{d,\varphi,\mathcal{V}} 0$ for every $j \in \mathbf{V}$, $K_{v,o} \xrightarrow{\mathcal{V}} 1$, then $\mathcal{T}_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in X_\varphi$.*

The proof analogous to that of Theorem 5 in [5] is omitted.

Now, let us denote: $\overline{x}_{j,K_v} = \{\underbrace{0, \ldots, 0}_{j-times}, K_{v,0}, K_{v,1}, \ldots\}$.

Theorem 4.5. *Let the assumptions of Lemmas 1 and 3 hold. If $\overline{x}_{j,K_v} \xrightarrow{d,\varphi,\mathcal{V}} 0$ for every $j \in \mathbf{V}$, then $\mathcal{T}_v(F) \xrightarrow{d,\varphi,\mathcal{V}} \mathbf{0}$ for every $F \in X_\varphi$.*

The proof analogous to that of Theorem 6 in [5] is omitted.

5. $P_k(Y)$-linear functionals

Now we will present some generalizations on the spaces of multifunctions of the classical Riesz theorems about a linear and continuous functional on a Banach space. We use the results of [6, 9].

Let now φ be an N-function and $\mathbb{N}$ be the set of all natural numbers. Let $\|\cdot\|_\varphi$ denote the Luxemburg norm in l^φ and $\|\cdot\|_\varphi^O$ denote the Orlicz norm in l^φ.

Definition 5.1. *The mapping $M : X_\varphi \to P_k(Y)$ such that $M(F+G) = M(F) + M(G)$, $M(aF) = aM(F)$ for all $F, G \in X_\varphi$, $a \geqslant 0$, will be called an $P_k(Y)$-linear functional on X_φ.*

Definition 5.2. *We say that $M : X_\varphi \to P_k(Y)$ is continuous at $F \in X_\varphi$ if for every $\epsilon > 0$ there is $\delta > 0$ such that from $\|\operatorname{dist}(F(\cdot), G(\cdot))\|_\varphi < \delta$ it follows that* $\operatorname{dist}(M(F), M(G)) < \epsilon$.

If M is continuous at every $F \in X_\varphi$, then we say that M is continuous on X_φ.

Let $f = (f_n)$, where $f_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. Denote

$$M_f(F) = \sum_{n=1}^{\infty} f_n F(n)$$

for every $F \in X$.

Lemma 5.3. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f = (f_k)$ where $f_k = 0$ for $k = n+1, n+2, \ldots$ Then M_f is a $P_k(Y)$-linear and continuous functional on X_{l^p}.*

Proof. We have

$$M_f(F) = \sum_{k=1}^{n} f_k F(k)$$

for every $F \in X_{l^p}$, so $M_f(F) \in P_k(Y)$ for every $F \in X_{l^p}$ and M_f is a $P_k(Y)$-linear. We also have for all $F, G \in X_{l^p}$ that

$$\operatorname{dist}(M_f(F), M_f(G)) \leqslant \sum_{k=1}^{n} |f_k| \operatorname{dist}(F(k), G(k)) \leqslant \|f\|_{l^q} \|\operatorname{dist}(F(\cdot).G(\cdot))\|_{l^p}.$$

□

Theorem 5.4. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in l^q$. Then M_f is a $P_k(Y)$-linear and continuous functional on X_{l^p}.*

Proof. Let $f = [f_1, \ldots, f_n, \ldots]$, $f_n = [f_1, \ldots, f_n, 0, 0, \ldots]$, $f \in l^q$.

It is easy to prove that for every $F \in X_{l^p}$ the sequence $\{M_{f_n}(F)\}$ is a Cauchy sequence in $\langle P_k(Y), \operatorname{dist}\rangle$, so there is $A \in P_k(Y)$ such that $\operatorname{dist}(M_{f_n}, A) \to 0$ as $n \to \infty$.

We also have for every $F \in X_{l^p}$:

$$\operatorname{dist}(M_f(F), M_{f_n}(F)) \leqslant \sum_{k=n+1}^{\infty} \operatorname{dist}(f_k F(k), \theta) \leqslant$$

$$\leqslant \sum_{k=n+1}^{\infty} |f_k||F(k)| \leqslant (\sum_{k=n+1}^{\infty} |f_k|^q)^{\frac{1}{q}} \||F|\|_{l^p} \to 0$$

as $n \to \infty$. So $M_f(F) = A$.

Let $F, G \in X_{l^p}$. We have

$$\begin{aligned}\mathrm{dist}(M_f(F), M_f(G)) &\leqslant \\ &\leqslant \mathrm{dist}(M_f(F), M_{f_n}(F)) + \mathrm{dist}(M_{f_n}(F), M_{f_n}(G)) + \mathrm{dist}(M_{f_n}(G), M_f(G)),\end{aligned}$$

so M_f is $P_k(Y)$-linear and continuous functional on X_{l^p}. □

Analogously we obtain the following two theorems (see also [9], Theorem 13.18):

Theorem 5.5. *Let $f \in m$. Then M_f is a $P_k(Y)$-linear and continuous functional on X_{l^1}.*

Theorem 5.6. *Let φ and its complementary $\varphi^\star$ be the N-functions, $\varphi = (\varphi_i)$, such that for every $u_0 > 0$ there is $c > 0$ for which $\frac{\varphi_i(u)}{u} \geqslant c$ for $u \geqslant u_0$ and $i \in \mathbb{N}$, (δ_2) holds for φ, $f \in l^{\varphi^\star}$. Then M_f in $P_k(Y)$-linear and continuous functional on X_φ.*

Denote

$$w_f(F) = \sum_{n=1}^{\infty} f_n |F(n)|,$$

for every $F \in X_{l^p}$.

Applying the proof of Theorem 5.4 we obtain the following

Theorem 5.7. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in l^q$, $f(n) \geqslant 0$ for every $n \in \mathbb{N}$. Then:*

$$w_f(F+G) \leqslant w_f(F) + w_f(G) \quad \text{and} \quad w_f(aF) = a w_f(F) \quad \text{for all} \quad F, G \in X_{l^p}.$$

Moreover, for every $\epsilon > 0$ there is $\delta > 0$ such that from $F, G \in X_{l^p}$, $\||F| - |G|\|_{l^p} < \delta$ it follows $|w_f(F) - w_f(G)| < \epsilon$.

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